Excellence in Mathematics

Report from the Maths Excellence Group

Foreword

The adoption of Curriculum for Excellence represents a significant step forward for education in Scotland. But it is only a step, and the biggest challenge lies ahead, because the curriculum that matters is not what is in the documents—it is in the lived experience of children and young people in Scottish schools.

Too often in the past, curriculum reform has been regarded as a simple process. It is not, because the curriculum comes to life only when each teacher finds a way of making sense of it in the context of her or his own classroom. Implementing a new curriculum is therefore not a process of giving new instructions to teachers. It involves every teacher in creating new knowledge about how to engage children and young people in new activities, new ideas and new kinds of learning. It is fundamentally a creative process.

This means that teachers, working together in schools, will need to explore, experiment, and exchange ideas about how to make Curriculum for Excellence work for them. No amount of well-meaning advice by experts can replace this.

So why have we written this report? Because while we cannot provide an instruction manual or a roadmap for implementing Curriculum for Excellence, we believe that we can help teachers avoid some of the potential blind alleys and dead ends, and provide some prompts that may stimulate reflection.

We have drawn on research evidence where appropriate, but there are many areas where the research evidence is inconclusive, or even contradictory, and teachers will need to use their professional judgment in deciding what weight to place on the evidence in this report.

There is much work to do, but the benefits are significant—to help children and young people in Scotland to understand and value the power of mathematics, to equip them with the skills to contribute effectively in the world of work and in civil society, but also, and perhaps most importantly, to exercise greater control over their own lives.

Dylan Wiliam, Co-chair, Mathematics Excellence Group
In my business career I have seen the relevance and application of mathematics skills in two distinct areas:

- as a business leader in the financial sector I have seen the importance of numeracy for all employees who work in our business – whether they are customer service staff, specialist finance or actuarial professionals, IT workers or HR and training staff

- through my work in the field of financial capability in schools and with people in later life I have seen the importance of numeracy in coping with the countless financial decisions we make every day.

Whether it is by owning a mobile phone for the first time, or saving for a new PC game, or football strip with the new star’s name emblazoned on the back – dealing with financial matters creates an opportunity to engage all children with mathematics in a relevant and exciting way.

I commend the work of the Excellence Group in producing such a readable and valuable report. It will contribute in a real way to improving the development of mathematics teaching in Scotland.

Otto Thoresen, Chief Executive, AEGON UK
Excellence in Mathematics

Introduction

‘To face the challenges of the 21st century, each young person needs to have confidence in using mathematical skills, and Scotland needs both specialist mathematicians and a highly numerate population.’ (Building the Curriculum 1)

Mathematics permeates all aspects of modern life. Mathematics continues to grow in importance in the world of work, but mathematical capability has also become essential for improving one’s personal life. Fostering excellence in mathematics is therefore not just vital to every nation’s economic prosperity. It is also essential to creating an equitable society.

Numeracy is an important part of mathematical capability, but mathematics is more than just becoming familiar and fluent with numbers. Mathematical capability includes:

- Using and applying skills in the real world, including the appropriate use of information and communications technology (ICT)
- Being open to new ideas and alternatives, and appreciative of the importance of evidence, and critical reasoning
- Being curious, imaginative and diligent

The sections that follow in this report will offer some details on and exemplars of key elements of excellence in mathematics, which will include:

- Motivating, engaging and imaginative use of mathematics across learning, in life and in work
- Characteristics of effective mathematics teaching and learning experiences.
- A clear and full understanding of essential knowledge and of progression within the development of skills and concepts in mathematics, including key numeracy skills and techniques.
- Effective assessment as the bridge between teaching and learning
- Maintenance and development by educators of essential mathematical knowledge and skills, and of appropriate pedagogical skills through planned, high quality continuous professional development (CPD)
- The importance of partnerships, external insights and support

Each section concludes with some questions for teachers, for leaders and for policymakers to generate reflection on the issues raised.
**Imaginative use of mathematics across learning, in life and work**

*Curriculum for Excellence* promotes the development of and support for Numeracy Across Learning as the responsibility of all teachers. Where appropriate, there should also be opportunities to apply mathematical concepts and skills in other curriculum areas and particularly in science, technology and engineering (together with mathematics these are often called the STEM disciplines). Successfully doing so highlights their significance and relevance, deepens learning across the curriculum, reinforces the numeracy and mathematical concepts and skills learned and, therefore, has the potential to strengthen the learner’s interest, motivation, engagement and achievement. To ensure depth, young people must be able to apply the key mathematical skills and understanding they have acquired in new, non-routine and relevant contexts. Central to this is the development of higher-order thinking skills that enable the learner to identify which particular mathematical techniques can be appropriately applied in order to progress towards a solution to a problem. When a new skill has been acquired, a degree of repeated practice and consolidation may be required in the short term but it is vital that learners are provided with a range of realistic opportunities and activities within which to apply the new skill in both familiar and unfamiliar contexts. For example, well-planned interdisciplinary activities that extend mathematical skills and allow their application in fresh, relevant, exciting and increasingly complex contexts will serve to engage and enthuse learners.

**Biomass Boiler, MTH 3-20b, TCH 3-06b, TCH 3-08b, MNU 3-20a**

Collaboration between Mathematics, Geography and Science

The school undertook an inter-disciplinary activity, centred on the ‘big question’ “What will the impact of future climate change be on the school’s current energy management system?”, provided by their Biomass Boiler. The project could reflect global issues through exploration in a local context and would demonstrate that numeracy skills are transferable across curricular areas.

Within mathematics, pupils used weather data from a local Met Office station, to plot a time series of average temperatures or rainfall against the energy output from the boiler in order to try to identify any correlation between the data sets. In Science, pupils visited the biomass boiler to find out more about it and they carried out a practical investigation into the power output from woodchips and looked at the cost of the energy. In Geography, pupils used weather graphs to discuss long-term trends, to make individual predictions about future trends and to discuss the impact of future trends on the energy cost for the school.

Throughout the activities, there was a focus on both numeracy skills and transferable life skills such as investigation, analysis and evaluation. As well as collaborating effectively to integrate numeracy learning across learning, teachers worked hard to ensure that concepts specific to the individual disciplines were also developed effectively.
In order to provide well-planned, imaginative and effective interdisciplinary working and collaboration

Teachers should consider:

- When planning activities, which features of interdisciplinary learning and collaboration they are emphasizing, and how this will add value in terms of its impact on young people’s learning?
- Do the activities provide children and young people with opportunities to talk about and explain their mathematical ideas and understanding?
- How they will promote the learner’s active participation and ensure that the learning provides appropriate challenge and skills progression?
- Whether the activities provide opportunities for learners to work independently and with others?
- In the course of the activities, are opportunities to make clear reference to key concepts maximised?
- Have the activities been designed to provide opportunities for good use of ICT to promote understanding and to enliven teaching?
- What steps have been taken to ensure that the activities are reviewed, improved and sustained?

School leaders should consider:

- How to ensure that there are structured and regular opportunities for key staff to collaborate to plan learning opportunities across learning, rich in number and mathematical contexts.
- How to develop a shared understanding and agreement among all teachers of key numeracy concepts, definitions, terminologies and teaching methodologies. This is essential to ensure that learners can make the desired connections in their mathematical learning across the curriculum areas.
- The implications for school systems and resources of enabling high quality interdisciplinary learning and collaboration.

Policymakers should consider:

- How can the assessment systems in place provide clear incentives for teachers and leaders to give appropriate emphasis to high quality interdisciplinary learning and collaboration.
The mathematical learning experience

A recent report from Her Majesty’s Inspectors of Education, Learning Together: Mathematics, suggested that effective learning in mathematical activities has the following characteristics:

**Good pace and depth of learning**
Teachers need to plan mathematical tasks and activities that meet the needs of all learners well. Key features of well-paced lessons include a clear and shared focus on the purpose and expected outcomes of learning, an optimum amount of time spent on active learning (rather than, for example, working through numerous pages in a textbook), high levels of learner motivation and engagement in thinking. Effective teaching ensures also that there will be no significant gaps in learners’ knowledge or skills development. To achieve an appropriate pace, teachers need to consider both the rate of learning and learners’ capacity to acquire new concepts. It should be noted that pace is not the same as speed. Pace in lessons requires that as much of the available time in lessons is spent with learners “minds on”—slower pace of learning but with a greater focus on learners’ deeper understanding can increase achievement. It is also important that artificial ceilings are not placed on the expectations of learners: some learners simply need a bit more time to understand fully important concepts and skills, or have them presented in a slightly different way.

**Challenge, enjoyment and positive attitudes**
Teachers should aim to set tasks and activities of increasing levels of complexity that continually develop, reinforce and extend learners’ understanding. Appropriate challenges and embedded problems will help learners make connections between aspects of mathematics and to their prior mathematical understanding. Activities should also promote higher order thinking skills including analysis, reasoning, deduction, and creativity (see Bloom’s revised taxonomy) and should require learners to apply their knowledge and skills to solve relevant problems; bringing the real world into the mathematics classrooms helps learners to make important connections so that they can use and apply the mathematics they learn in the classroom elsewhere.

**Independent and collaborative learning**
Independent learning requires learners to be confident enough to make smart choices, for example, about the resources they use, how to approach tasks and in terms of monitoring and evaluating their own progress. Effective collaborative learning challenges individuals to think independently as well as to share their thinking, to engage actively in discussion and to debate and learn from each other.

**Active Learning**
One of the most important things learned about the human brain in the 20th century was that learning is not a process of passively acquiring knowledge—we actively construct our knowledge, and the more engaged we are in activities, the more likely they are to result in lasting learning. This does not mean that children and young people need to be physically active while learning. Rather it means that teachers should provide opportunities for play and other activities which require learners to engage cognitively with, and respond to their learning, to analyse and make decisions, explain their thinking, and synthesise aspects of their existing skills. Providing knowledge and skills without due reference to other connected aspects of mathematics or relevant contexts and meaningful activities will limit learners’ mathematical development. Learners should be encouraged to think deeply about mathematical ideas and concepts, to
develop and deepen their own understanding and to use their existing skills and knowledge in different contexts, including their application in unfamiliar and non-routine problems.

Confidence and security
It is important that learners are secure in their mathematical knowledge and understanding. They need to be able to recognise the mathematics in a range of contexts and have the confidence to apply the skills. To do this, learners need a good range of rich experiences. But they also need to understand that everyone can learn mathematics. Many adults (including parents) say that they were “no good at maths”. Many adults appear to be happy to admit this, while for others, it is a real source of anxiety.

In a report for the Scottish Government

III, Diana Coben described how many adults looking back at their mathematical life histories shared similar experiences. Three themes were particularly common:

The ‘brick’ wall – the point (usually in childhood) at which mathematics stopped making sense; for some people it was long division, for others fractions or algebra, while others never hit the brick wall. For those who did, the impact was often traumatic and long-lasting.

The ‘significant other’ – someone perceived as a major influence on the person’s maths life history. The influence might be positive or negative, past or present. Significant others included, for example, a parent who tried to help with maths homework; a teacher who made the person feel stupid; a partner who undermined the person’s confidence in their mathematical abilities.

The ‘door’ – marked ‘Mathematics’, locked or unlocked, which people have to go through to enter or get on in a chosen line of work or study.

For some, the emotional reaction can be so profound as to induce feelings of dread that interfere strongly with the concentration needed to carry out mathematical tasks

IV and the assumption that ‘I can’t do maths’ takes over. While practice can be important in addressing this, it is usually much more productive to alleviate anxiety and to support the learner, for example by using alternative approaches, and making sure that learners do not feel unduly pressured to complete activities quickly. Helping learners shift from “I can’t do this” to “I can’t do this—yet” builds confidence that effort generates success in mathematics, and it can also be helpful to support parents and carers so that they too can support learning. Teachers and tutors may well become ‘the significant others’ for a number of children and young people. Whether that significance is positive or negative depends entirely on the teacher and their teaching methods. They might be the person who helps to unlock the door or who gives a pupil the confidence to pass through it, for example by using alternative approaches, and making sure that learners do not feel unduly pressured to complete activities quickly.
In order to generate engaging and effective learning experiences for mathematics

Teachers should consider:

- Which of the current aspects of their practice are most effective in achieving the kinds of mathematical experiences described in this section?
- How can they build on these successes to apply these approaches in other areas of their practice?

School leaders should consider:

- What kinds of support will teachers need in developing their practice in the way described in this section?
- How can the best balance be struck between accountability and support to ensure that teachers continue to meet the needs of the learners while also taking measured risks on improving their practice?

Policymakers should consider:

- How can what is known about effective practice in mathematics teaching be more effectively communicated to practitioners and leaders?

Characteristics of effective teaching in mathematics

The HMIe report Learning together: mathematics notes that effective teaching in mathematics requires teachers to increase children’s ability to use mathematics to solve problems. However, not all teachers identify appropriate and relevant contexts, instead relying too much on commercially produced schemes. In order to achieve higher standards in mathematics and to make it more relevant to learners, continuous professional development (CPD) must feature more strongly within schools and local learning communities, whereby staff can share good practice, model effective teaching, coach/mentor and provide CPD opportunities for colleagues.

Connecting learning in numeracy and, where appropriate mathematics, with that in other curriculum areas (particular those related to the STEM disciplines) through well-planned interdisciplinary learning will help ensure appropriate challenges and progression in skills development within enriched, authentic contexts. Many concepts in mathematics are inter-related, so knowing one helps you understand others. Learners need to be given opportunities to experiment with the concepts that they learn, to apply them to other areas, to reformulate them and describe them to someone else. A number of studies of effective teaching of mathematics\(^{v}\) have found it is important to emphasise connections:

- between different aspects of mathematics, such as connections between operations (e.g., that addition is the inverse of subtraction) and, connections between topics such as shape,
number (and algebra) such as those generated by investigating the number of lines needed to create a ‘mystic rose’ inside the 5-point circle below:

**Number** of lines = 4 + 3 + 2 + 1 = 10 (Why?)

**Formula** = \( \frac{1}{2} n (n-1) \), where \( n = 5 \) points

\[ \frac{1}{2} \times 5 \times 4 = 10 \]

This example can also be used to underscore the importance of going beyond just finding patterns and understanding the reasons behind them, by looking at the number of regions created within the circle by the addition of extra points:

<table>
<thead>
<tr>
<th>Number of points</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of regions</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>?</td>
</tr>
</tbody>
</table>

The pattern suggests that 6 lines will create 32 regions but the answer is actually 31 regions. This example highlights the importance of such mathematical concepts as testing conjectures and proof.

- between different representations of mathematics. This could include moving between symbols, words, diagrams, objects and graphs;

- with learners’ methods. This could include valuing their methods, being interested in their thinking and sharing their methods.

Teaching approaches should consistently involve sharing the purpose of learning with children, encouraging their engagement in tasks, in evaluation of their progress and achievement, and in using high quality feedback to inform planning of next steps in learning.

Teachers should use a range of techniques to elicit students’ understanding. Questioning in classrooms should have two main aims: to cause thinking, and to provide evidence about current achievement that teachers can use to adapt their teaching to better meet the needs of learners. While open-ended questions are essential to effective teaching, carefully-designed closed questions can be valuable too. Questions like “Can a triangle have two right angles?” are clearly closed (there are two answers, and one of them is wrong!) but if students do not know the answer, then such a question can provoke valuable discussion in the mathematics classrooms. It is also worth noting that sometimes, it can be more effective to make statements rather than asking questions. The question “Are all squares rectangles?” is likely to elicit either a bald “yes” or “no” whereas making the statement “All squares are rectangles” seems to encourage students to provide reasons for their agreement or disagreement, and thus enrich the quality of classroom dialogue.
Teachers should also listen interpretively, rather than evaluatively to learners’ responses. When students answer questions incorrectly, it is often possible, by exploring the reasons for the answer, to establish why the student responded incorrectly. If all the teacher discovers is that the student does not know the correct answer, the only recourse for the teacher is to teach the material again.

Effective teaching also necessitates maximizing the opportunities for children to work collaboratively. Interacting with others to share understanding, develop and explain thinking, and to explore and demonstrate learning ensures that learners are engaged and active in classrooms.

In short, effective teaching places learners at the centre of their own learning, provides opportunities for learners to make choices about resources, methodology and evaluation of their progress and achievement, encourages shared, high expectations in respect of effort, appropriate progress for each learner and achievement, and will ensure that all learners feel valued and supported sufficiently to seek help as they need.

**In order to secure effective teaching of mathematics**

**Teachers should consider:**

- What are the key aspects of the development and progression of mathematical skills and concepts?
- What are the key building blocks in mathematical learning and where to go back to if a child is struggling to go forward?
- How can they continue to maintain and develop their mathematical knowledge and teaching methods?
- How can they sustain high levels of expectations of learners’ effort and engagement, progress and achievement, and thus motivate learners to do even better?

**School leaders should consider:**

- How they can create opportunities for teachers to share their knowledge and to learn for each other.

**Policymakers should consider:**

- What support and incentives are needed for teachers and leaders to create the kinds of learning environments for teachers that will be necessary for the continual improvement of practice?

**Essential knowledge for teaching mathematics**

Mathematics is a sequential subject; the learning achieved at each stage of development of a skill provides the foundation for learning at the next stage. Each stage of development, however, benefits from being reinforced through challenging practice, including through contextualised problems and applications across learning, in real life and in the workplace. Therefore, as well as
making progress through the stages of a mathematical concept or skill, a learner can also make progress in their capacity to apply that skill in an appropriate way.

Some learners make smooth, even progress across the range of mathematical skills being developed but for most, this will not be the case. Many encounter “stumbling blocks” that appear insuperable, but in retrospect, the learner cannot understand what was so difficult. Most learners have more aptitude in some skills than in others, and the vast majority of learners progress in “fits and starts” rather than smoothly (the Leverhulme Numeracy Research Programme found that only 10% of primary school children made steady progress over a four year period—the other 90% had at least one six-month period where they made no progress, or went backwards in their learning of mathematics).

Teachers of numeracy and other aspects of mathematics must be confident in their understanding and competent in their application of key concepts and skills and how they can interconnect. Key elements of essential concepts and skills, some of which provide challenges in teaching, include:

- **Knowledge and understanding of number, the number system and its operations.** It is vital that children from the earliest stages of mathematical learning develop a sound sense of counting and a clear understanding of how numbers relate to size, quantity and order. A ‘sense of number’, has been defined as ‘... an intuitive understanding of numbers, their magnitude, relationships, and how they are affected by operations.’vi Having a well-developed sense of number allows an individual to manipulate numbers, comprehend the connections between operations, decipher written ‘problems’ and tackle calculations that have a number of steps. This can be a challenging aspect of mathematics to teach and it cannot be achieved by learners doing pages of written sums. The freedom to ‘play around’ with numbers without worrying about exact answers is one way to tackle this. Marvin Minsky—one of the leading cognitive psychologists of the last half century—suggests that one of the key problems is that mathematics teaching focuses too much on preventing mistakes and too little on promoting inventiveness in learnersvii

- **Well developed skills in calculating mentally.** Developing a good range of flexible strategies for calculating mentally enhances learners’ progress and achievement in mathematics. Some commonly used algorithms like ‘decomposition’ for subtraction prove challenging for many learners to master. By ensuring that learners become adept at using such skills as ‘counting on or back from’, teachers can provide them with alternative, more straightforward strategies which are often more successful. Teachers should also encourage learners to understand and use inverse processes to simplify a problem. Subtracting 29 from 42 is much more difficult if ‘decomposition’ is used when compared with “counting on” (i.e., working out what needs to be added to 29 to get 42). Simple representations such as the ‘empty number line’ or using a ‘rekenrek’ (developed by the Freudenthal Institute in the Netherlands) can also greatly assist learners to calculate effectively.
• **Understand and using inverse processes to simplify a problem.** For many learners, calculating $32 \div 4$ can be daunting, but is much simplified on appreciating the relationship between $32 \div 4$ and $8 \times 4$. In many countries, where levels of numeracy are high, teaching mental calculation forms the basis of early learning in number which is complemented by the introduction of formal algorithms at appropriate stages. Learners will make better progress in their mathematics if they can manipulate appropriate number facts accurately and speedily and apply them within real situations involving eg money, measure and time, which will help them become more confident individuals. Whilst it is important that learners are able to recall and apply facts speedily and accurately, it should be noted that skills learned in rote fashion will atrophy whereas skills acquired with deeper conceptual understanding will be more easily reacquired, as and when necessary.

• **Skills in estimating appropriate quantities.** Estimation is something that learners often find difficult. This may be as a result of an over-emphasis on getting “the right answer” but of course, being able to decide if an answer is approximately correct is vital even where accuracy is important. Teachers need to develop in their students not just the skill of estimating, but also the ability to decide how accurate an answer needs to be, for a given context.

• **Recognising, working with, extending, and justifying patterns based on attributes and numbers.** Children and young people are often fascinated by patterns, including number patterns. A key skill that underpins mathematical and numerical development is the ability to spot a pattern, to continue and extend the pattern, to discover a rule governing it, and to provide a mathematical justification of why the pattern does, indeed, continue in the way described. The ability to spot number patterns is enhanced by the knowledge of the most commonly encountered patterns in mathematics, such as the square numbers, the triangle numbers and the [Fibonacci sequence](https://en.wikipedia.org/wiki/Fibonacci_number).

• **A sound grasp of the concept of place value.** Understanding place value is absolutely essential to a firm grasp of number. Initial development based on the context of money enables learners to relate place value to other contexts. The use of appropriate concrete resources will help learners to picture place value “in action” and improve their understanding. One particularly powerful way of doing this is by using the idea of bundles of 10 sticks, and bundles of bundles, as in the following representation of the number 234:
The appropriate use of ICT (e.g. interactive white boards) can provide a dynamic context for learning together with helpful imagery, which can enhance the learning experience and aid longer-term understanding.

- A confidence in using appropriate mathematical language and notation. The language of mathematics can be confusing to learners. For example, the terms ‘subtract, ‘minus’, ‘take away’ and ‘difference between’ all relate to the same process but many learners do not easily relate all these terms to that process. Children need to be introduced from early stages to and gain confidence in using mathematical language, to continue to develop that language appropriately throughout their learning and to use it confidently when explaining their thinking and answers to others. There are commonly used terms that have a particular meaning in mathematics lessons that learners need to recognise and understand such as the product of two numbers, the mean of a set of data and the derivative of a function;

- The application of key mathematical properties and relationships. Such properties include the commutative, associative and distributive properties, and the concept of ‘variable’, an understanding of which provide for an effective progression from specific number work into more generalised number and the skills of algebra; for example, the commutative property for addition, exemplified by $3 + 5 = 5 + 3$, later generalises to $x+y = y+x$. A robust understanding of the concept of inverse relationships will support effective calculating strategies. As noted above, the problems $8 \times 4$ and $32 \div 4$ are interconnected, and are related to $\frac{1}{4}$ of 32. Familiarity with, and practice of, these kinds of re-expressions will provide a route towards robust understanding of more abstract relationships, such as algebraic inverse functions. Initially the inverse processes met are:
  - addition with subtraction
  - multiplication with division.
An understanding of the inverse nature of these processes is the foundation for solving equations and it also highlights inverse relationships throughout mathematical learning such as
  - squares with square roots
  - removing brackets with factorisation
  - enlargement with reduction
  - exponential functions with logarithmic functions
• Understanding and applying the concepts, notation and related operational skills of fractions, decimal fractions and percentages (and their interrelationships) and to be able to apply these skills in learning, life and the workplace. Teachers can assist learners’ understanding of these concepts by ensuring that they are given a range of representations of fractions from the outset. For example, fractions can be shown not only showing fractions as segments of a ‘pizza’ or ‘cake’ or as part shadings of an area but also as points on the number line. Teachers need also to emphasise the relationship between fractions, division and ratio. A popular resource, Thinking Through Mathematics, recommends that educators attempt to create connections between topics as learners often find it difficult to generalise and transfer their learning to other topics and contexts. Effective teachers build bridges between ideas.

• Successful progression from specific numbers to generalised algebraic thinking. Learners need to progress from competence with number processes and associated inverse operations to the use of letters (or symbols) in simple formulae and in simple mathematical expressions to represent a variable, where any one of a variety of numbers can be used. This is the stepping stone to mathematical modelling using relationships such as $F = \frac{9}{5} C + 32$ (converting temperature from Centigrade to Fahrenheit) and analysis, usually involving the solving of equations. Teachers need to be aware of the confusion that can arise when difference disciplines use the same symbol for different quantities. For example, it is common to use the letter $s$ for speed in the mathematics classroom, whereas in the physics classroom, it is often used to denote distance. Raising such matters with students as potential difficulties increases the likelihood that students check the meaning of a symbol rather than assuming it means the same thing in all contexts.

• An ability to classify and relate 2D shapes and 3D objects using their key properties. Programmes in mathematics often highlight recognition of common shapes and objects but do not always place sufficient emphasis on their properties. It is vital that learners understand the importance of investigating appropriately such properties as edges, faces, vertices, angles, symmetry, area and volume. Encourage learners to ‘play’ with shapes and objects by building and drawing them, comparing them (similar and congruent shapes), folding them (symmetry) and turning them (angle). Working with geometrical shapes (such as the square and the circle) to understand their basic features is necessary before any in depth study of their properties can be meaningful. This leads in turn to analysis of, and calculations within, complex mathematical diagrams, usually involving angles for which the groundwork done over earlier stages of development will now also come into play.

• Being able to solve a wide range of problems, originating from real life or from within mathematics learning or from scenarios encountered across the curriculum areas. As learners progress through their mathematics programme, such problems will grow in complexity and levels of interpretation and will mature in relation to the contexts involved. Solving problems is an ideal environment in which children and young people can apply the mathematics they are learning. In primary schools in Singapore, for example, a structured approach is taken to developing skills in solving word problems, based on the number of steps involved in solving the problem and effective ‘bar-modelling’, shown below, is used to provide learners with a visual representation of a problem to be solved.
Learners can apply this simple pictorial model to a wide range of mathematical problems.

In order that those involved in the teaching of mathematics have the knowledge they need

Teachers should consider:

- What specific topics present particular difficulties for students, and how can they find time to work with other practitioners to identify, and solve, common problems?

School leaders should consider:

- Whether the existing time available for administration and professional development within the school can be used differently in order to enable teachers to share practice and plan collaboratively?

Policymakers should consider:

- How can the progression between the initial preparation of teachers and their continuing professional development be aligned to ensure that teachers develop, both in pre-service and in-service training, the knowledge they need to maximize learning?

Assessment as the bridge between teaching and learning

Learners do not learn what we teach. Or to put it more carefully, it is not possible to predict with any certainty what learners will learn as a result of any particular sequence of classroom activities. That is why the most effective teachers spend a considerable amount of time finding out what students have learned before they attempt to teach anything else. And that, in turn, is why assessment is at the heart of effective teaching.
Of course, there is a role for assessment conducted at the end of a sequence of classroom activities, so that the teacher can record what the students have learned. However, this kind of assessment at the end of the learning process rarely influences what happens next. For assessment to be at its most powerful, the assessment must take place before the end of the learning, so that the information from the assessment can be used to adjust the classroom activities to better meet student learning needs.

This role of assessment is sometimes called ‘formative assessment’, since the assessment forms the direction of future learning. How often, and in what way the assessment is conducted, is far less important than the idea that the assessment should be used to help the teacher make better decisions about what do next than he or she would have been able to make without the evidence from the assessment. Here are some examples.

1. A teacher plans a three-week unit on fractions and decimals. Since the class spends an hour on mathematics every day, she has fifteen hours to spend on the unit. However, she plans to cover all the material in the unit over the first twelve days. On the thirteenth day, the students in the class do a test on what they have been learning. The teacher collects in the students’ test papers, but does not mark them. Instead, she reads the students’ papers, and on the basis of what she sees, she plans what she is going to do with the class on days fourteen and fifteen.

2. A teacher has been teaching students about lines of symmetry, and ten minutes before the end of the lesson, she displays the following images on the board:

   ![Images of shapes A, B, C, D, E, F]

   She calls out “A” and each student in the class has to vote by holding up the right number of fingers to show how many lines of symmetry shape A has, and then does the same for B, C, D, E and F. As the students vote, she doesn’t try to remember how each student responds. Instead she focuses on just two things:

   - Are there any items that a significant proportion of the class answer incorrectly, and which she needs to re-teach with the whole class?
   - Which two or three students would benefit from individual attention?

She sees that many students indicate that they think that shape D has just three lines of symmetry, and that shape E has no lines of symmetry even though it is, in fact, a rhombus. She points out to the class that D has six lines of symmetry, three through the corners, and three
through the mid-points of the edges and reminds the class to look for lines of symmetry that are not horizontal, vertical, or at 45°. In both of these cases, she realizes that the incorrect answers may not necessarily indicate poor mathematical understanding. After all, in the case of shape D, holding up just three fingers could be a result of students thinking that only one hand would be necessary, and as a result of the distortion of the data projector, the image of shape E might not actually be the rhombus shape intended. Nevertheless, she is well aware that in teaching, it is better to assume students do not know something when in fact they do than to assume that the do know something, when in fact, they don’t.

3. A teacher has been teaching graph-sketching to an S5 class. They appear to understand the important ideas, but just to be sure, she asks every student in the class to sketch, on their mini-white boards, “the graph of \( y = \frac{1}{x} \).” The students hold up their responses, and she sees that they all are able to do this. She moves on to the next chapter.

In the first two of these cases, the assessment was carefully planned, but in the third it was done “on the fly”. In the first of these cases, the length of the cycle was three weeks, while in the third, it was two minutes. The common feature is that in each case, the teacher collected high-quality information about student achievement in order to make a decision about what to do next.

Many teachers in Scotland will by now be convinced of the benefits of using formative assessment in their classrooms, others will have read the literature, attended in-service days, possibly even experimented a little, but are still doubtful of its value.

The Mathematics Excellence Group advocates strongly the planning of questions into lesson preparation. Such questions have been called ‘hinge questions’. The idea is that the teacher plans every lesson with a ‘hinge’; a point in the lesson when the teacher can check on student understanding, and then decide what to do next. ‘Hinge’ questions are typically designed to test learners’ understanding of one important concept in a lesson—one that is critical for pupils to comprehend before the teacher moves on in the lesson. However, since it is meant to be a quick check on learning, rather than a learning activity in itself, each student should respond to the question within two minutes (and preferably less) and the teacher must be able to collect and interpret the responses from all students in 30 seconds, whether by ‘finger voting’, classroom clickers, mini white boards, or some other system.

**In order to ensure that assessment is used to support learning, as well as measure it**

**Teachers should consider:**

- What specific techniques they can begin to use to improve the quality of evidence they generate for the decision they need to make during teaching.

**School leaders should consider:**

- How can senior leaders monitor and evaluate the development of teachers’ assessment skills, and share good practice across the school?
Policymakers should consider:

• How can the assessments generated by teachers in different schools be evaluated and shared, to reduce the extent to which every teacher or school has to “reinvent the wheel”?

Continuous professional development (CPD)

No matter how well it is designed, initial preparation cannot possibly meet all teachers’ professional learning needs. Career-long professional learning is not, therefore, a ‘nice to have’ but an essential part of professional practice, both to improve outcomes for learners, and to improve teacher retention. At present, the majority of site-based professional development activities include peer coaching, mentoring, modelling, observing and providing feedback to others. It is most likely to be enhanced in a lasting way when CPD policy and practice allow for richer learning experiences than are usually offered in short courses based on a developmental model of skills acquisition.

Contemporary thinking on CPD indicates that it is most effective when it is aligned with existing school culture, is led by peers, is collaborative and is sustained—for more than one term and ideally for more than a year. A review of experimental studies found that one-off CPD sessions and programmes of less than 14 hours in total showed no statistically significant effect on student learning, while programmes with between 30 and 100 contact hours over six to 12 months showed a significant positive effect on student outcomes. The authors concluded that “sustained and intensive professional learning for teachers is related to student-achievement gains”\textsuperscript{x}.

Other reviews of research have also provided strong evidence for the effects of CPD. One review\textsuperscript{xi} identified four sets of factors that appeared to be particularly important for professional development to be effective. First, teachers need to have some control over their professional development, so that it could meet the changing needs of teachers as they gained experience. Second, the professional development must be focused on the outcomes for learners—professional development that is not, at least in some way, designed to improve outcomes for learners is self-indulgent. Third, effective professional learning required the support of senior leaders within the school. Fourth, professional learning is considerably enhanced by external support and networking, with other schools, with local authorities, and with higher education institutions. Another review\textsuperscript{xii} looked systematically and in detail at the contents of CPD programmes that increased student achievement. While the timescales, patterns of meetings, and the amount of support varied, there were a number of common features:

• the CPD built on what teachers knew and could do already, with an emphasis on individual learning
• teachers were encouraged and guided in supporting each other
• there was some external input, in terms of both theory and practice, for example by sharing experiences from other schools and teachers, or from research studies
• there was a shift in the “ownership” of the reform from those providing the external stimulus to the teachers themselves.
Although none of the three reviews discussed above focused specifically on the teaching of mathematics, it would appear that positive outcomes for the learner and teacher will result from a longer term commitment to the development of teachers’ skills throughout their careers. There must also be an alignment between the professional needs and school and national developments so that there is a greater awareness of how to identify training needs, in turn leading to more tailored and effective CPD provision. CPD is most effective when it involves a cycle of reflection and discussion between practitioners and the outcomes are integrated into classroom practice.

Support for CPD is available from local, peer ‘coaches’ and ‘mentors’, from Education Authority providers and from national providers such as Learning & Teaching Scotland, Tapestry, the Scottish Mathematical Council and through CPD Find.

In order to benefit from well-planned CPD which will provide development in both subject-based knowledge and teaching skills

**Teachers should consider:**

- what their personal CPD needs are.
- how these needs can best be met through a sustained programme of development.
- how they can maintain and develop their mathematical knowledge and pedagogical skills as their career develops.

**School leaders should consider:**

- how best to ensure that the development needs of staff are identified.
- how these needs align with school and national developments.
- how best to engage the staff in planning a programme that will allow these needs to be met, using local, peer support and other providers.
- the implications for the school of decisions taken relating to CPD provision.

**Policymakers should consider:**

- what kinds of national frameworks will be most effective in ensuring that teachers receive useful and appropriate professional development throughout their careers.

**External insights, partnerships and support**

When the impact and relevance of modern education is discussed in the national media the provocative question "what is the point of studying mathematics?" is often raised. Carl Friedrich Gauss, who was labelled the "Prince of Mathematicians", himself described mathematics as the "Queen of the Sciences", reflecting its fundamental role underpinning a huge range of fields in the physical, biological and social sciences, as well as engineering, communications, transport,
commerce, economics and medicine. In addition, even the most esoteric aspects of pure mathematics may find their influence – as exemplars of precise and logical thinking – in fields such as law, ethics and philosophy. In view of its influence on so many aspects of our modern society the relevance of studying mathematics would seem beyond doubt. How, then, can mathematics teachers in Scottish schools best harness the remarkable wider societal impact of their subject to enthuse and inspire students? An excellent range of external resources and agencies exists to help.

Science Connects (www.scienceconnects.org.uk) seeks to promote and coordinate outreach in STEM subjects, linking schools with local University and industry contacts, who are trained as STEM Ambassadors. Science Connects also provides learning resources and regular news updates for schools, highlighting opportunities for involvement in practical STEM activities. Schools can thus access in a variety of ways the knowledge and enthusiasm of experts who are employing mathematics in their studies, their research or their careers. Links with schools can range from a single-visit lecture or workshop, led e.g. by one or more graduate students, to an extended collaboration. In the latter case several Scottish universities offer a “Schools’ Ambassador” course to their senior undergraduates: here a student spends a few months working closely with a single teacher or class, helping to deliver a series of lessons on a particular topic. In a related initiative, STEM-ED Scotland has recently launched “Connecting it up”, a project aiming specifically to integrate support for Curriculum for Excellence across STEM disciplines. This project is developing learning progressions that unite the STEM subjects around ‘big ideas’ in science and mathematics – recognizing that these can be taught more effectively through cross-disciplinary collaboration and reinforcement. The importance of mathematics to the Sciences was highlighted by the Deans of Science who stated that, ‘Using these (mathematical) skills in the course of science study should significantly enhance the science itself, whilst also reinforcing progress in mathematics.’ Key mathematical skills which are important within STEM disciplines include scale, proportion, measurement, data handling and probability, geometry and graphs, handling equations and algebra, manipulating units as well as a facility with the concepts of justification and proof. At more advanced levels such skills as trigonometry, coordinate geometry, vectors and calculus are key. The inclusion of mathematics within the Scottish Science Baccalaureate it also welcomed and by building strong mathematical skills in the CfE Broad General Education and into the Senior Phase, this would also enhance achievement in the STEM disciplines.

Scottish industry can also directly provide strong exemplars of the far-reaching influence of mathematics and its importance for future career prospects; these may be particularly effective if they are drawn from within the local community. CBI Scotland and local Chambers of Commerce can assist teachers with establishing contacts. A number of our larger businesses and companies – for example in engineering, electronics, IT and the financial sector – already have staff dedicated to schools liaison and outreach, but for Scotland’s many SME’s the opportunity for establishing schools’ links may be seen as no less important and mutually beneficial.

In the wider media the recent growth of high-profile science programming on television, particularly on the BBC, has included excellent examples in pure and applied mathematics: “The Story of Maths”; “The Beauty of Diagrams”; “Beautiful Equations”; “The Joy of Stats”; “The Secret Life of Chaos”; “Wonders of the Solar System”. These programmes are accessible and entertaining while offering deep insights into the influence and wonder of mathematics, and are presented by passionate communicators who stand as powerful role models for aspiring mathematicians and scientists.

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In order to maximise the potential benefits to be gained from effective external contacts, partnerships and support

Teachers should consider:
• how to best utilise available external contacts, partners and support to help make learning more relevant and engaging, and to enable learners to make connections to the real world by applying their mathematical learning.

School leaders should consider:
• how best to facilitate and sustain such external links and partnerships so that they will flourish and impact positively on learning.

Policymakers should consider:
• in what ways they can best promote and further develop opportunities for effective partnerships working e.g. through local and national clubs, events and competitions.

References


vi Definition from Learning NC accessed online on 5/1/11 at http://www.learnnc.org/reference/number+sense See also Understanding a Child’s Development of Number Sense at http://www.illuminations.nctm.org/Reflections_preK-2.html

vii http://wiki.laptop.org/go/What_makes_Mathematics_hard_to_learn%3F


